THE NUMBERS OF FACES OF SIMPLICIAL POLYTOPES

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ABSTRACT

In this paper is considered the problem of determining the possible f-vectors of simplicial polytopes. A conjecture is made about the form of the solution to this problem; it is proved in the case of d-polytopes with at most $d+3$ vertices.

1. Introduction

In recent years much attention has been focussed on extremal problems involving the numbers of faces of convex polytopes. For example, one consequence of the revival of interest in polytopes spurred on by questions raised in linear programming was the formulation by Motzkin [13] of what has come to be known as the Upper-bound Conjecture. This conjecture, recently proved by the author [10] concerns the maximum possible number of faces of a polytope of a given dimension with a given number of vertices.

The corresponding minimal problems appear to be more difficult, and it is only with the restriction to the class of simplicial polytopes that significant progress has been made. Such progress includes the proofs of the so-called Lower-bound Conjecture by Walkup [15] in dimensions 4 and 5, and Barnette [1] in the case of facets, and the formulation by McMullen-Walkup [12] of a generalization of the Lower-bound Conjecture.

In this paper we shall discuss the most general problem of finding all the possible sequences of numbers of faces of simplicial polytopes. Perhaps because no conjectures about the form of the answer have, so far, been made, little progress has been made on this problem. Here we shall formulate such a conjecture, and prove it in case of polytopes with few vertices.

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2. The conjecture

Let P be a simplicial d-polytope, and for $j = 0,\cdots,d-1$, let $f_j(P) = f_j$ be the number of its j-faces. (For the terminology used in the paper, the reader should consult Grünbaum [3].) The sequence $(f_0,\cdots,f_{d-1})$ is called the *f-vector* of P. We shall adopt the conventions $f_{-1} = 1$, and $f_j = 0$ if $j < -1$ or $j \geqq d$. It is a familiar fact that the numbers f_j satisfy a set of linear equations, known as the *Dehn-Sommerville equations* ([2; 14]; see also Klee [5], Grünbaum [3, §9.2]). These are, for $-1 \leqq j \leqq d-1$,

$$\sum_{i=j}^{d-1} (-1)^i \binom{i+1}{j+1} f_i = (-1)^{d-1} f_j.$$

(Clearly the last equation, for $j = d-1$, is trivial; the first, for $j = -1$, is just Euler's relation; see Grünbaum [3, §8.1].) For any sequence $(f_0,\cdots,f_{d-1})$ we write

$$g_k = \sum_{j=-1}^{k} (-1)^{k-j} \binom{d-j}{d-k} f_j.$$

Then it can be shown ([12], where g_k is denoted $g_k^{(d+1)}$) that the Dehn-Sommerville equations are equivalent to the following equations: for $k = -1,\cdots,[\frac{1}{2}(d-1)]$,

$$g_k = -g_{d-k-1}.$$

The relationship between the numbers f_j and g_k can also be expressed in the form

$$\begin{aligned} f_j &= \sum_{k=-1}^{j} \binom{d-k}{d-j} g_k \\ &= \sum_{k=-1}^{n-1} \left\{ \binom{d-k}{d-j} - \binom{k+1}{d-j} \right\} g_k, \end{aligned}$$

where we adopt the notation (here and elsewhere) $n = [\frac{1}{2}d]$. Thus, the numbers f_j can be written as linear combinations of $g_{-1},\cdots,g_{n-1}$ with non-negative coefficients.

Let a and j be positive integers. We define the j-canonical representation of a as follows ([7]; see also Grünbaum [3, §10.1]). If the numbers $c_j, c_{j-1},\cdots,c_{m+1}$ have already been defined, and

$$a > \binom{c_j}{j} + \binom{c_{j-1}}{j-1} + \cdots + \binom{c_{m+1}}{m+1}$$

we define c_m by

$$c_m = \max\left\{ c \,\middle|\, a > \binom{c_j}{j} + \cdots + \binom{c_{m+1}}{m+1} + \binom{c}{m} \right\}.$$

The sequence $(c_j, c_{j-1}, \cdots)$ clearly terminates for some c_i with $i \geqq 1$, and the numbers $c_j, \cdots, c_i$ then satisfy

$$c_j > c_{j-1} > \cdots > c_i \geqq i.$$

The j-canonical representation of a is then

$$a = \binom{c_j}{j} + \binom{c_{j-1}}{j-1} + \cdots + \binom{c_i}{i}.$$

Now if $k > j$ is any other positive integer, we define the fractional power $a^{\langle k|j\rangle}$ by

$$a^{\langle k|j\rangle} = \binom{c_j + k - j}{k} + \binom{c_{j-1} + k - j}{k-1} + \cdots + \binom{c_i + k - j}{i + k - j}.$$

(Compare Kruskal [7], Grünbaum [3, §10.1].) We also adopt the natural convention $0^{\langle k|j\rangle} = 0$.

We are now able to state the conjecture which is considered in this paper.

CONJECTURE. *The sequence $(f_0, \cdots, f_{d-1})$ is the f-vector of some simplicial d-polytope if and only if*

$$g_k = -g_{d-k-1} \qquad (k = -1, \cdots, [\tfrac{1}{2}(d-1)])$$

$$g_k \geqq 0 \qquad (k = 0, \cdots, n-1),$$

$$g_k \leqq g_{k-1}^{\langle k+1|k\rangle} \qquad (k = 1, \cdots, n-1).$$

(We remind the reader that $n = [\tfrac{1}{2}d]$, and that the convention $f_{-1} = 1$ $(= g_{-1})$ is used throughout.)

3. Some remarks on the conjecture

We first observe that, since every d-polytope has at least $d+1$ vertices, the inequality $g_0 \geqq 0$ always holds. Further, a d-polytope with v vertices has at most $\binom{v}{2}$ edges. This implies

$$g_1 \leqq \binom{v-d}{2} = g_0^{\langle 2|1\rangle}.$$

The conjecture has the following implications. Let us write $f_0 = v$ for the number of vertices. Then the inequalities $g_k \leqq g_{k-1}^{\langle k+1|k\rangle}$ imply that, for $k = 1, \cdots, n-1$,

$$g_k \leqq g_0^{\langle k+1|1\rangle} = (v-d-1)^{\langle k+1|1\rangle} = \binom{v-d+k-1}{k+1}.$$

These inequalities yield corresponding upper-bounds for the numbers f_j. For $j = 1,\cdots, n-1$, we have

$$f_j \leqq \sum_{k=-1}^{j} \binom{d-k}{d-j}\binom{v-d+k-1}{k+1} = \binom{v}{j+1},$$

with equality if and only if the polytope is $(j+1)$-neighbourly; that is, every subset of $j+1$ vertices is the set of vertices of a j-face. For $j = n,\cdots, d-1$, the coefficient of each g_k in the expression for f_j given in the previous section is positive. Thus, the upper-bound for f_j is only attained when each g_k attains its maximum, and so the polytope must be n-neighbourly. Bearing in mind the fact that a d-polytope with v vertices and the maximum possible numbers of faces must be simplicial (see Klee [6], Grünbaum [3]), McMullen [9], we see that our conjecture implies the Upper-bound Conjecture, which we know to hold [10].

The inequalities $g_k \geqq 0$ $(k = 0,\cdots, n-1)$ are just those of the Generalized Lower-bound Conjecture for simplicial polytopes, formulated by McMullen-Walkup [12]. It was further conjectured in that paper that equality $g_k = 0$ holds only for polytopes which admit a subdivision into a simplicial d-complex, every $(d-k-1)$-face of which is a face of the polytope. This would naturally imply that $g_j = 0$ for each $j = k+1,\cdots, n-1$, an implication which would also follow from our conjecture.

McMullen-Walkup [12] showed by means of examples that any linear inequality satisfied by the numbers f_j is a consequence of the inequalities $g_k \geqq 0$ $(k = 0,\cdots, n-1)$. However, the inequalities of our conjecture are not all linear, and would imply that not all possible sequences $(f_0,\cdots,f_{d-1})$ lying in the n-dimensional simplicial cone determined by the Dehn-Sommerville equations and the McMullen-Walkup inequalities correspond to simplicial d-polytopes.

4. The cases $v \leqq d+2$

We shall now establish the conjecture in the cases $v \leqq d+3$ (where $v = f_0$). We begin with the easier cases $v \leqq d+2$.

The only d-polytopes with $d+1$ vertices are the d-simplices, for which

$$f_j = \binom{d+1}{j+1} \qquad (j = -1,\cdots d-1),$$

and so

$$g_k = 0 \qquad (k = 0,\cdots n-1).$$

This case clearly satisfies the conditions of the conjecture.

In the case $v = d + 2$, since $g_0 = 1$, and $1^{\langle k+1|k\rangle} = 1$ for all $k = 1, 2, \cdots$, the conjecture is equivalent to: for some $k = 1, \cdots, n$,

$$g_j = 1 \qquad (j = 0, \cdots, k - 1),$$

$$g_j = 0 \qquad (j = k, \cdots, n - 1).$$

Now the simplicial d-polytopes with $d + 2$ vertices are just the polytopes $T^{k,d-k}$ [11]; denoted T_k^d or T_{d-k}^d by Grünbaum [3, §6.1], for $k = 1, \cdots, n$. $T^{k,d-k}$ is the convex hull of a k-simplex and a $(d - k)$-simplex with a single relatively interior point of each in common. From the expressions for the numbers of faces of $T^{k,d-k}$ [3, Theorem 6.1.2], it may be seen that the corresponding numbers g_j are precisely those given above.

5. The case $v = d + 3$

We first reformulate the conditions of the conjecture. As we have already observed, the conditions of the conjecture imply that, for $k = 1, \cdots, n - 1$,

$$0 \leqq g_k \leqq k + 2.$$

We have two possibilities, according to the k-canonical representation of g_{k-1}. If $g_{k-1} = k + 1$, it is clear that the full range of possible values for g_k, namely that given above, is allowed. On the other hand, suppose $g_{k-1} = m \leqq k$. If $m > 0$, the k-canonical representation of m is

$$m = \binom{k}{k} + \binom{k-1}{k-1} + \cdots + \binom{k-m+1}{k-m+1},$$

and so the range of values allowed for g_k is

$$\begin{aligned} 0 \leqq g_k \leqq \; & m^{\langle k+1|k\rangle} \\ = \; & \binom{k+1}{k+1} + \binom{k}{k} + \cdots + \binom{k-m+2}{k-m+2} \\ = \; & m. \end{aligned}$$

If $m = 0$, then $g_k = 0$ also. In other words, we have

$$0 \leqq g_k \leqq g_{k-1}.$$

So, to verify the conjecture, it is enough to prove

THEOREM. *The sequence $(f_0, \cdots, f_{d-1})$ is the f-vector of some simplicial d-polytope with $d + 3$ vertices if and only if, for some $k = 1, \cdots, n$,*

$$g_j = j + 2 \qquad (j = 0, \cdots, k-1),$$

$$0 \leqq g_j \leqq g_{j-1} \qquad (j = k, \cdots, n-1).$$

We shall prove the theorem using the technique of Gale diagrams ([3, §§5.4 and 6.3]; McMullen-Shephard [11, Chapter 3]). A standard Gale diagram of a simplicial d-polytope P with $d + 3$ vertices is a subset $\hat{P}$ of $d + 3$ points (not necessarily distinct) of the unit circle S in the plane E^2, such that at least two points of $\hat{P}$ lie on each side of every diameter of S, and no two points of $\hat{P}$ lie at opposite ends of any diameter of S. If the vertices of P are $x_1, \cdots, x_{d+3}$, and the corresponding points of $\hat{P}$ are $\hat{x}_1, \cdots, \hat{x}_{d+3}$, then the condition that (say) $x_1, \cdots x_{j+1}$ are the vertices of a j-face of P is just that

$$o \in \operatorname{int} \operatorname{conv} \{\hat{x}_{j+2}, \cdots, \hat{x}_{d+3}\},$$

where o is the centre of S.

A diameter of S which contains a point of P will be called a *diameter* of $\hat{P}$; points of $\hat{P}$ on a diameter L of $\hat{P}$ will be said to be *associated* with L. In an obvious way, we can talk about *adjacent* diameters of $\hat{P}$, and, since $\hat{P}$ must have at least three diameters, we may distinguish points of $\hat{P}$ associated with adjacent diameters as being at the *same* or *opposite* ends of those diameters.

It follows from the above description of $\hat{P}$ that we may rotate the diameters of $\hat{P}$ about o in any fashion, without changing the combinatorial type of the corresponding polytope P, as long as we do not allow adjacent diameters whose associated points are at opposite ends to cross. In particular, two adjacent diameters whose associated points are at the same end can be brought into coincidence, and a diameter with two or more associated points can be separated into the corresponding number of diameters with a single associated point, all of which are at the same ends. Performing these operations as many times as possible leads, respectively, to *contracted* and *distended* Gale diagrams of P. A contracted Gale diagram has an odd number of diameters, and points associated with adjacent diameters are at opposite ends; a distended Gale diagram has $d + 3$ diameters, each with a single associated point.

We shall, for the most part, work with distended Gale diagrams. The difference between the numbers of points of $\hat{P}$ on each side of a diameter L of $\hat{P}$ is called the *excess* of L. The first of the conditions on $\hat{P}$ implies that the maximum possible excess of a diameter is $d - 2$; note that the excess has the same parity as d. It can easily be verified that adjacent diameters of $\hat{P}$ whose associated points are at opposite ends have the same excess; if the associated points are at the same ends, then the excesses differ by 2, except that both excesses may be 1. By considering the way the excess changes as we move around o, we also see that if d is even $\hat{P}$ has at least one diameter of excess 0, and if d is odd $\hat{P}$ has at least two diameters of excess 1 (whose associated points are at the same end).

If we denote by p_e the number of diameters of $\hat{P}$ of excess e, we deduce from the above remarks that, if $e \geqq 4$ and $p_e > 0$, then $p_{e-2} \geqq 2$. So, if $d \geqq 2$ and $e \geqq 2$ (with the same parity), then

$$\sum_{j<e} p_j \geqq e - 1.$$

Since $\Sigma_j p_j = d + 3$, it follows that

$$\sum_{j \geqq e} p_j \leqq d - e + 4.$$

We next observe that if $e \geqq 2$ is the maximal excess of any diameter of $\hat{P}$, then the p_e diameters of excess e occur in adjacent pairs, whose associated points are at opposite ends (for, we observe that the diameters adjacent to the diameters of an odd number of adjacent diameters of excess e have excesses $e - 2$ and $e + 2$, contradicting the maximality of e).

We finally remark that if the maximum excess of a diameter of $\hat{P}$ is e, then the polytope P is $\frac{1}{2}(d - e)$-neighbourly, for at least $\frac{1}{2}(d - e) + 1$ points of $\hat{P}$ lie on each side of every diameter of S.

The proof of the theorem will depend upon the following operation on the distended Gale diagram $\hat{P}$. Let L and M be adjacent diameters of $\hat{P}$, whose associated points $\hat{x}$ and $\hat{y}$, respectively, are at opposite ends. The $d + 1$ remaining points of $\hat{P}$ fall into two subsets, say r points $\hat{z}_1, \cdots, \hat{z}_r$ such that

$$o \notin \operatorname{conv} \{\hat{x}, \hat{y}, \hat{z}_i\} \qquad (i = 1, \cdots, r),$$

and s points $\hat{w}_1, \cdots \hat{w}_s$ such that

$$o \in \operatorname{conv} \{\hat{x}, \hat{y}, \hat{w}_j\} \qquad (j = 1, \cdots, s)$$

(see Fig. 1 below). Thus $r + s = d + 1$. Note that the conditions on $\hat{P}$ imply that $r \geqq 1$ and $s \geqq 2$. Now suppose $r \geqq 2$ also. If we transpose the diameters L and M (as in Fig. 2), then we reverse the relations above, and therefore, in general, change the combinatorial type of the corresponding polytope P.

The corresponding change in the numbers of faces of P is easily calculated. For, a j-face of P corresponds to a subset of $d - j + 2$ points of $\hat{P}$, with o in their convex hull. Such subsets which contain at most one of $\hat{x}$ or $\hat{y}$ are not affected in this respect by the transposition, and so the changes arise from those containing both $\hat{x}$ and $\hat{y}$. Before the transposition, we obtain such a subset by choosing, apart from $\hat{x}$ and $\hat{y}$, $t \geqq 0$ points from $\hat{z}_1, \cdots, \hat{z}_r$ and $u \geqq 1$ points from $\hat{w}_1, \cdots, \hat{w}_s$, with $t + u = d - j$. Thus, the number is

$$\sum_{t+u=d-j, t \geqq 0, u \geqq 1} \binom{r}{t} \binom{s}{u}.$$

After the transposition, we must choose $t \geqq 1$ from $\hat{z}_1, \cdots \hat{z}_r$, and $u \geqq 0$ from $\hat{w}_1, \cdots, \hat{w}_s$, again with $t + u = d - j$. So, the number is

$$\sum_{t+u=d-j, t \geqq 1\ u \geqq 0} \binom{r}{t} \binom{s}{u},$$

and the resulting increase Δf_j in the number of j-faces is

$$\Delta f_j = \binom{r}{d-j} - \binom{s}{d-j}.$$

It follows at once that the corresponding increase Δg_j in g_j is

$$\begin{aligned} \Delta g_j &= \sum_{i=-1}^{j} (-1)^{j-i} \binom{d-i}{d-j} \left\{ \binom{r}{d-i} - \binom{s}{d-i} \right\} \\ &= \delta_{j\,d-r} - \delta_{j\,d-s}, \end{aligned}$$

when δ denotes the Kronecker delta function.

Now let e be the maximal excess of the diameters of $\hat{P}$. We have already observed that the polytope P is then $\frac{1}{2}(d-e)$-neighbourly, so that

$$f_j = \binom{d+3}{j+1} \qquad (j = -1, \cdots, \tfrac{1}{2}(d-e)-1),$$

or

$$g_j = j+2 \qquad (j = -1, \cdots, \tfrac{1}{2}(d-e)-1).$$

If $e = 0$ or 1, there is nothing more to prove, so suppose that $e \geqq 2$. As we have remarked, the p_e diameters of $\hat{P}$ of excess e can be grouped in adjacent pairs with their associated points at opposite ends. We may thus transpose these $\frac{1}{2}p_e$ pairs, in the manner just described. It is easy to verify that the values of r and s are given by

$$r + s = d + 1, \qquad r - s = e - 1,$$

so that

$$r = \tfrac{1}{2}(d + e), \quad s = \tfrac{1}{2}(d - e + 2).$$

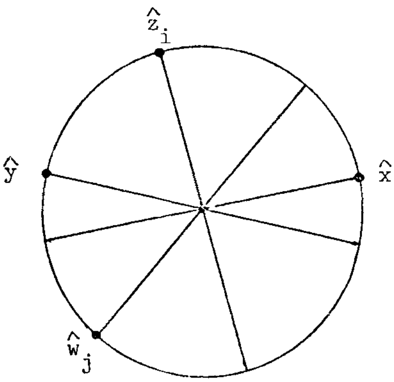

Fig. 1

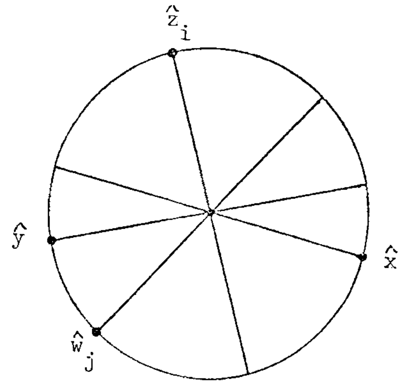

Fig. 2

Upon transposing any pair, the increase Δg_j in g_j is thus

$$\Delta g_j = \delta_{j.\frac{1}{2}(d-e)} - \delta_{j.\frac{1}{2}(d+e-2)}.$$

In other words, in the range $j = 0, \cdots, n - 1$, we see that $g_{\frac{1}{2}(d-e)}$ is increased by 1, while the other g_j's are unaltered. Hence if we transpose all the pairs of diameters of excess e, the total increase in g_j is 0, unless $j = \frac{1}{2}(d - e)$, when the increase is $\frac{1}{2}p_e$. But we observe that each transposition decreases the excesses of the diameters involved by 2, so that the resulting Gale diagram has diameters of maxima excess $e - 2$. Thus the new value of $g_{\frac{1}{2}(d-e)}$ is

$$\tfrac{1}{2}(d - e) + 2 = \tfrac{1}{2}(d - e + 4),$$

so that, for the original diagram $\hat{P}$, we had

$$g_{\frac{1}{2}(d-e)} = \tfrac{1}{2}(d - e + 4) - \tfrac{1}{2}p_e.$$

If $e - 2 \geqq 2$, we now apply the same argument to the new Gale diagram, noting that the number of its diameters of excess $e - 2$ is $p_e + p_{e-2}$. An easy induction argument then leads to the general expressions

$$g_j = j + 2 - \tfrac{1}{2} \sum_{s=1}^{j} p_{d-2s} \qquad (j = 0, \cdots, n - 1).$$

But we have already observed that $\sum_{j \geqq e} p_j \leqq d - e + 4$, and so, for $j = 1, \cdots, n-1$,

$$\begin{aligned} g_j &\geqq j + 2 - \tfrac{1}{2}(d - (d - 2j) + 4) \\ &= 0. \end{aligned}$$

We complete the first half of the proof of the theorem by remarking that if, for some $k = 1, \cdots, n-1$, $g_{k-1} \leqq k+1$, then $p_{d-2k+2} > 0$, so that $p_{d-2j} \geqq 2$ for each $j = k, \cdots, n-1$. Thus, for $j = k, \cdots, n-1$,

$$\begin{aligned} g_j &= j + 2 - \tfrac{1}{2} \sum_{s=1}^{j} p_{d-2s} \\ &= j + 1 - \tfrac{1}{2} \sum_{s=1}^{j-1} p_{d-2s} + 1 - \tfrac{1}{2} p_{d-2j} \\ &\leqq g_{j-1}. \end{aligned}$$

It only remains to show that any set of numbers g_j which satisfies the conditions of the theorem corresponds to some simplicial d-polytope with $d+3$ vertices. We shall construct a contracted Gale diagram of such a polytope.

If $g_j = j+2$ for $j = 0, \cdots, k-1$, but $0 \leqq g_j \leqq g_{j-1}$ for $j = k, \cdots, n-1$, the Gale diagram $\hat{P}$ will have $2k+3$ distinct diameters, which we label in cyclic order $L_0, L_1, \cdots, L_{2k+2}$.

Points of $\hat{P}$ associated with adjacent diameters are at opposite ends. If i is odd, then L_i has one associated point. If $l > 1$, then we associate with L_{2l} one more point than the number of $j \geqq k$ for which $g_j = l$. With L_0 are associated the remainder of the $d+3$ points of $\hat{P}$. Elementary (but tedious) calculations applied to a distended Gale diagram corresponding to $\hat{P}$ show that the numbers p_e of diameters of excess e satisfy

$$\sum_{s=1}^{j} p_{d-2s} = 2(j + 2 - g_j),$$

as required. This completes the proof of the theorem.

6. Final remarks

In dimensions 4 and 5, the only inequality of the conjecture which is not trivial is $g_1 \geqq 0$, or, equivalently,

$$f_1 \geqq d f_0 - \binom{d+1}{2} \qquad (d = 4, 5).$$

However, this is just the inequality of the Lower-bound Conjecture, proved by Walkup [15] in dimensions 4 and 5 (the proof by Barnette [1] also covers these cases). Implicit in the description by Grünbaum [3, §10.4] of the possible pairs (f_0, f_1) for 4-polytopes is a proof that f_1 achieves every value such that

$$4f_0 - 10 \leqq f_1 \leqq \binom{f_0}{2}.$$

or

$$0 \leqq g_1 \leqq \binom{f_0 - 4}{2} = g_0^{\langle 2|1\rangle}.$$

Similar constructions will show that the range $0 \leqq g_1 \leqq g_0^{\langle 2|1\rangle}$ is also achieved by simplicial 5-polytopes, and since the cases $d \leqq 3$ are trivial, this shows that the conjecture of the paper holds in dimensions $d \leqq 5$.

In fact Walkup [15] proved that, if $d = 4$ or 5, the f-vectors of triangulations of the $(d-1)$-sphere are precisely those which satisfy $0 \leqq g_1 \leqq g_0^{\langle 2|1\rangle}$. Further, Mani [8] has shown that every triangulated $(d-1)$-sphere with at most $d+3$ vertices is isomorphic to the boundary complex of some (simplicial) d-polytope. Thus, in every case in which the conjecture is known to be true, it also holds for the corresponding triangulated spheres. We might therefore be led to suppose that the conjecture should properly apply to triangulated spheres, rather than to simplicial polytopes. However, there are fundamental differences between triangulated $(d-1)$-spheres and boundary complexes of simplicial d-polytopes. For example, if $d \geqq 4$, Mani's result is false for $(d-1)$-spheres with $d+4$ vertices (see Grünbaum [3, §11.5]; these differences are also discussed in Grünbaum [4]). We should therefore, perhaps, be wary of extending the conjecture to triangulated spheres.

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